Visualization of linear transformations and eigenvectors in 2D and 3D

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Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the standard basis of $\mathbb{R}^2$.

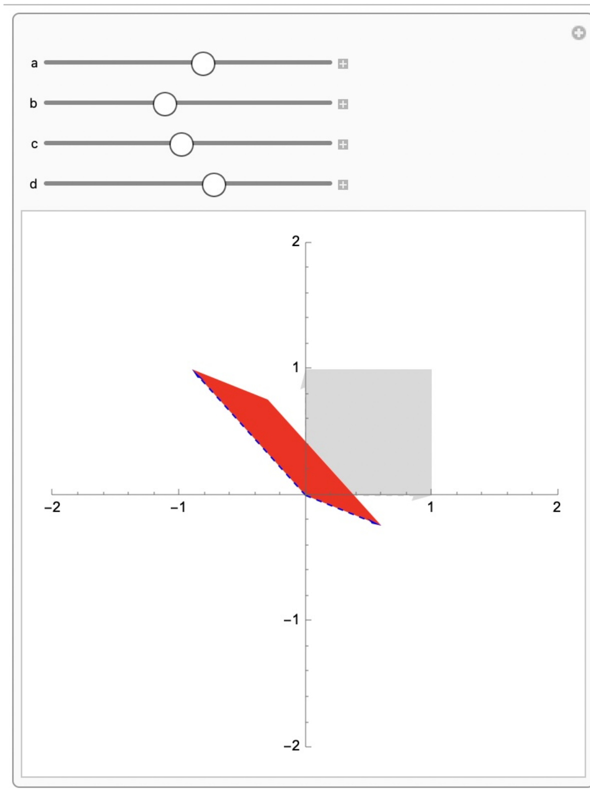
The images show how the vectors in unit square in $\mathbb{R}^2$ transform under the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for different values of a, b, c, d.

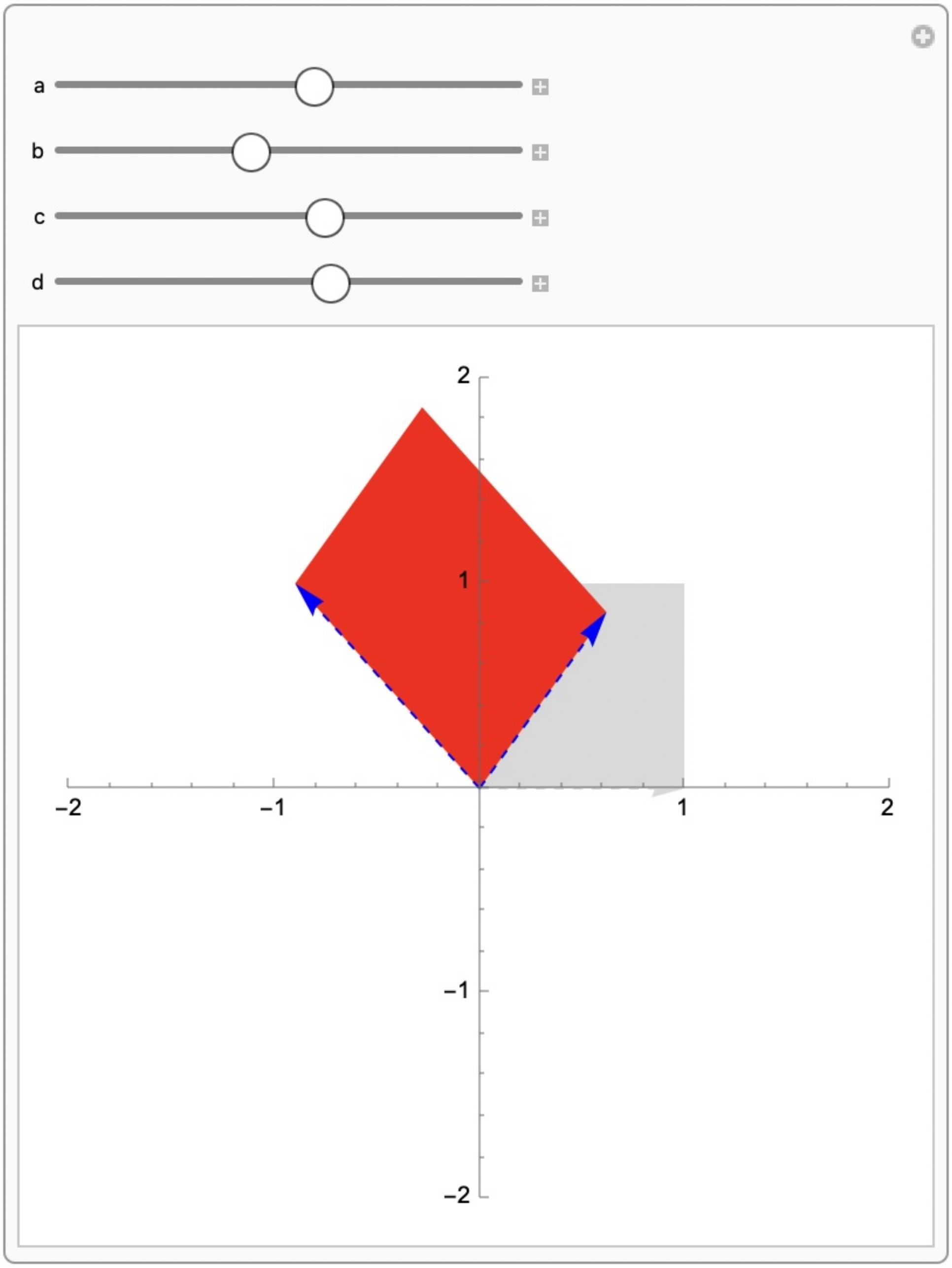
The grey region shows the unit square in $\mathbb{R}^2$.

The dashed black lines indicate the standard basis vectors.

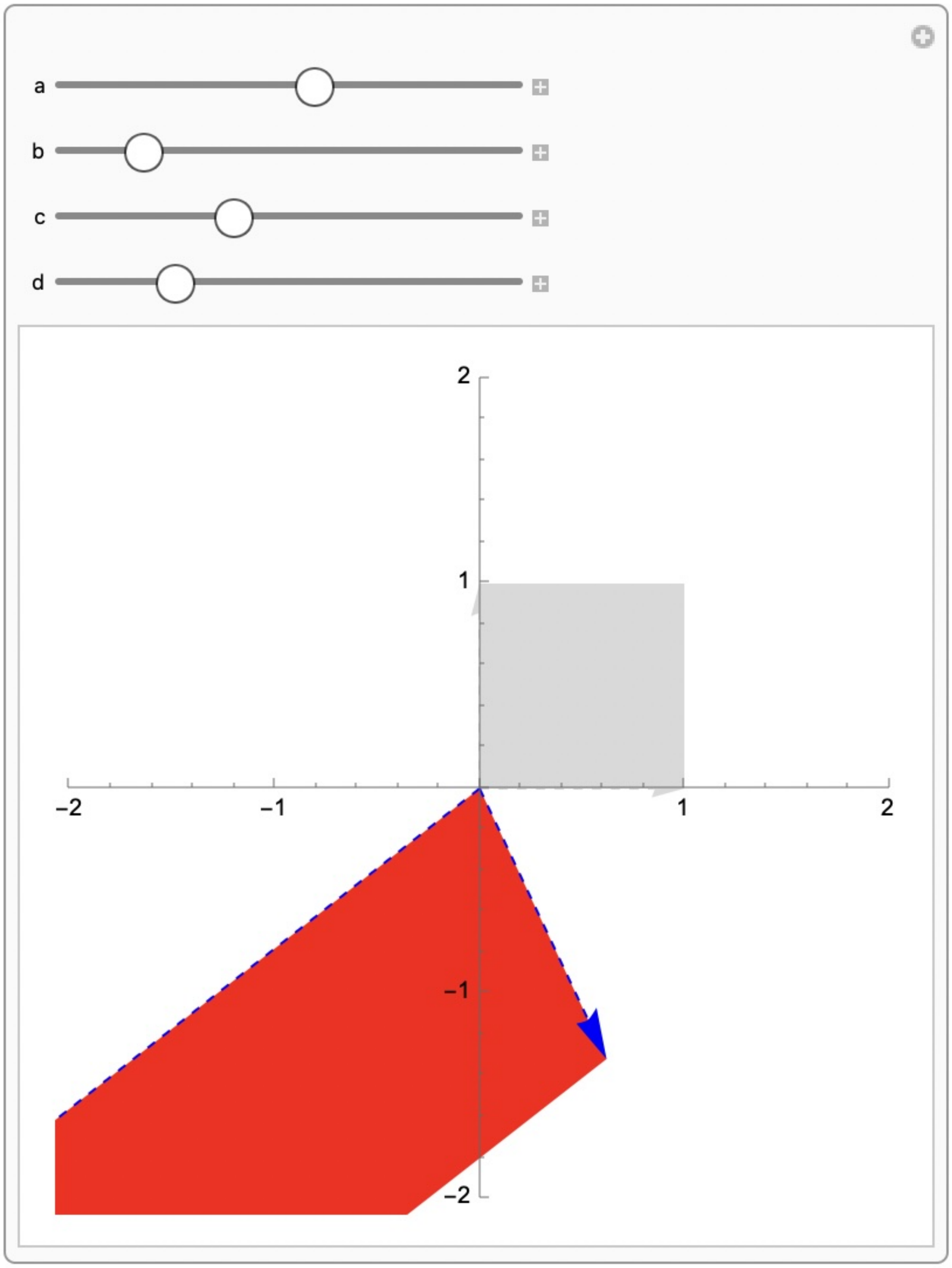
The red region shows the transformed unit square.

The blue vectors indicate the transformed basis vectors.



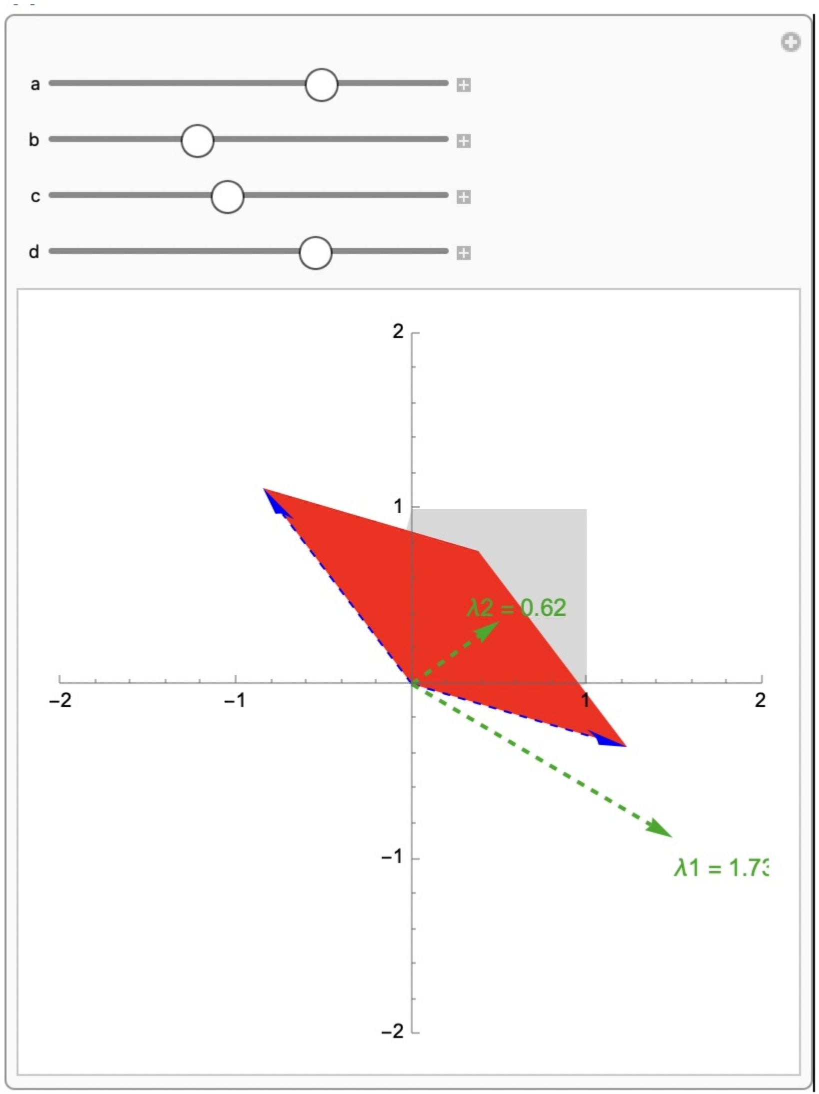


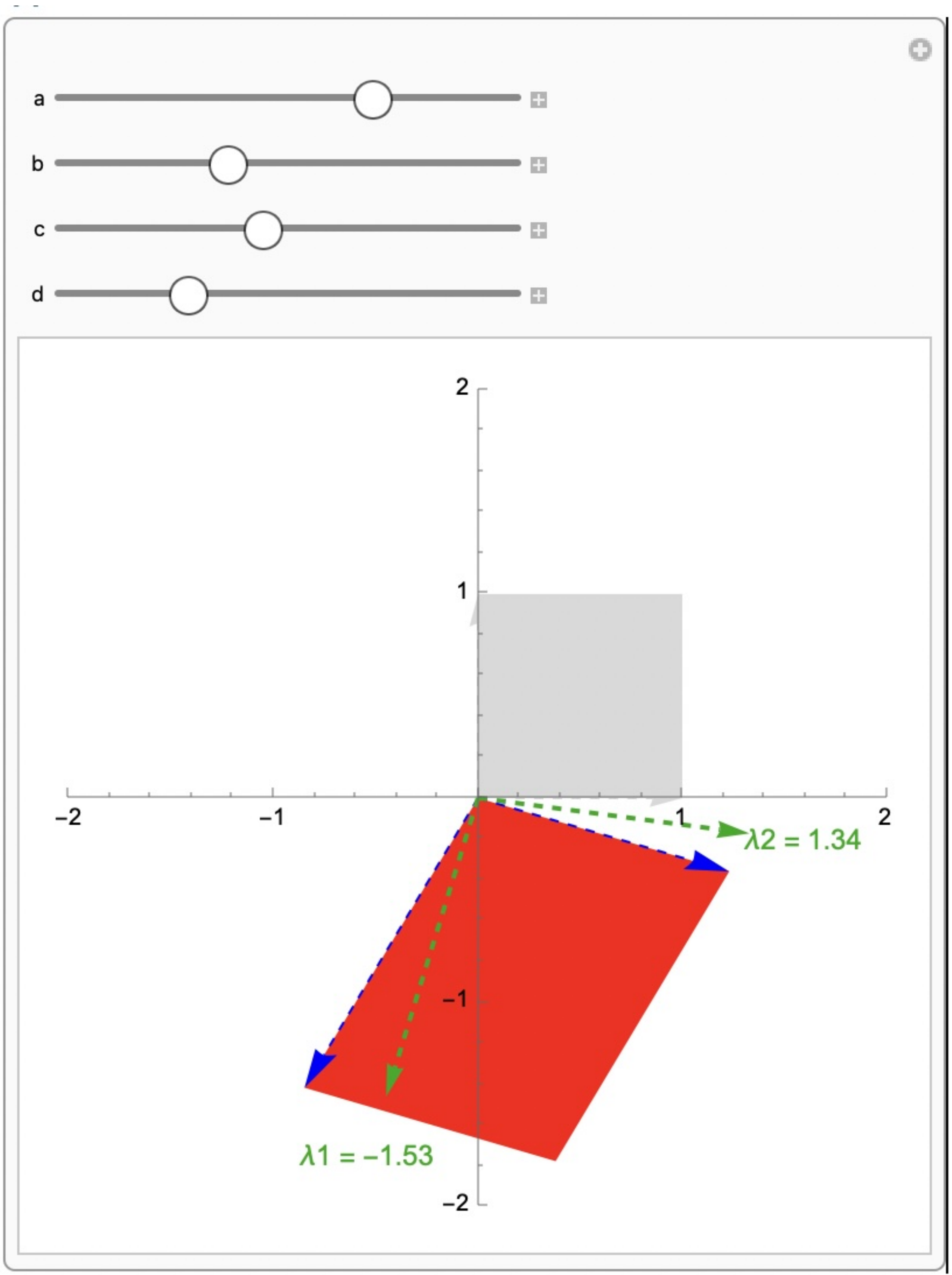
a
b
c
d
2
1
−1
−2
−2
−1
1
2


a
b
c
d
2
1
−1
−2
−2
−1
1
2

Eigenvectors of linear transformations in $\mathbb{R}^2$

The following image depicts the eigenvectors for the linear transformation with matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for fixed a, b, c, d. The blue vectors indicate the transformed basis vectors, the red region shows the transformed unit square, the green arrows indicate the eigenvectors, labeled with eigenvalues. The eigenvectors do not change direction under the transformation.




a
b
c
d
λ2 = 1.34
λ1 = −1.53

Example: A diagonal matrix

A diagonal matrix represents a linear transformation that scales each coordinate axis independently. Let

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

This matrix acts on a vector $\mathbf{x} = (x, y) \in \mathbb{R}^2$ by scaling the x-component by λ_1 and the y-component by λ_2:

$$A\mathbf{x} = \begin{pmatrix} \lambda_1 x \\ \lambda_2 y \end{pmatrix}.$$

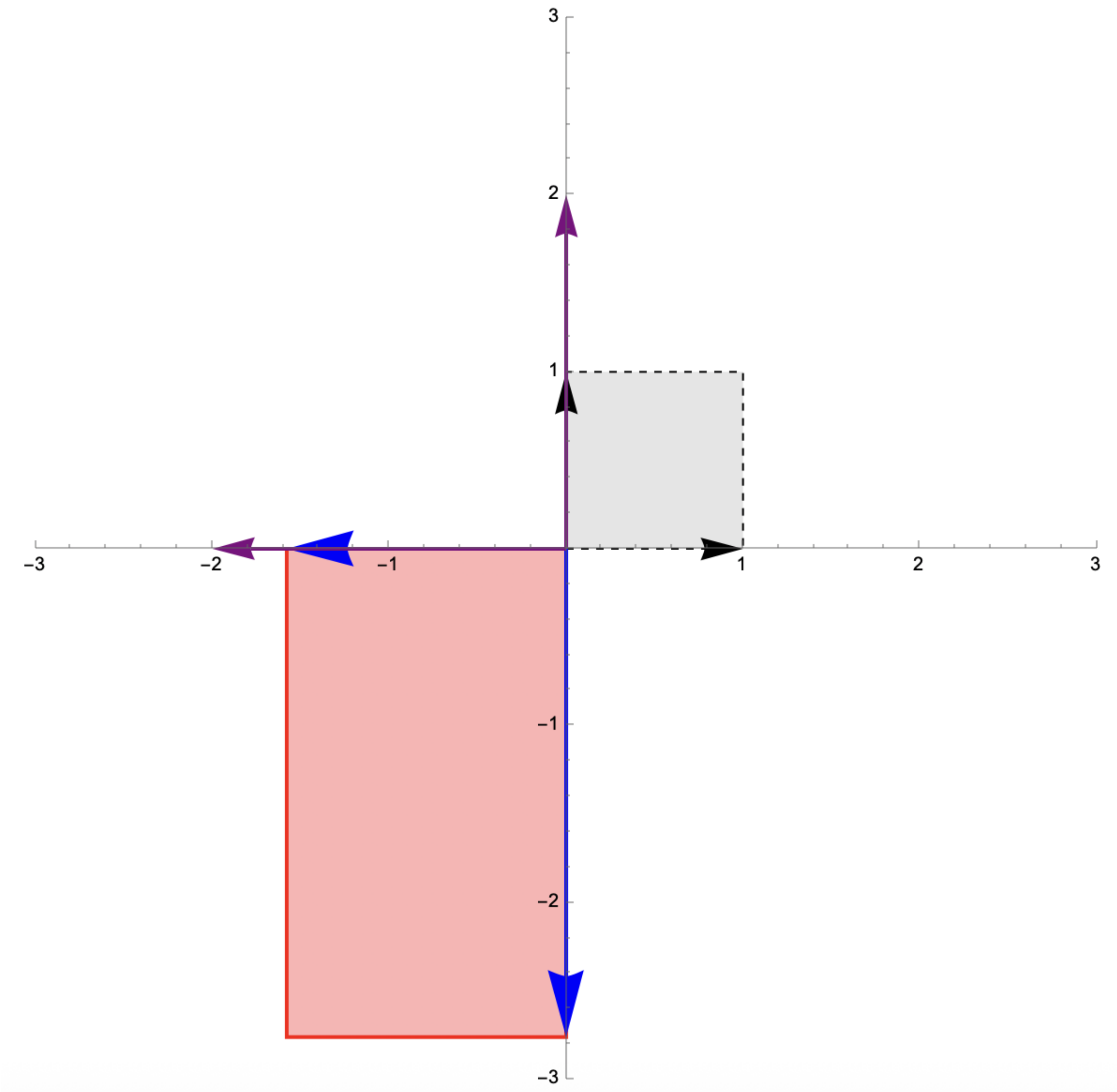
The eigenvectors of A are the standard basis vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

corresponding to the eigenvalues λ_1 and λ_2, respectively.

If $\lambda_i > 1$, the transformation expands along the i-th axis; if $0 < \lambda_i < 1$, it contracts along that axis; and if $\lambda_i < 0$, it reflects and scales along that axis.

The gray square represents the original unit square in the plane. The red square shows the image of the unit square after transformation by A. The black dashed black arrows indicate the original basis vectors. The blue arrows show the transformed basis vectors. The purple arrows represent the eigenvectors of A, which coincide with the coordinate axes.


3
2
1
−3
−2
−1
1
2
3
−1
−2
−3

Example: A 2×2 Jordan block

A Jordan block represents a linear transformation that combines scaling and shearing. Let

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The parameter λ scales all vectors by the same factor, while the upper off-diagonal entry 1 introduces a shear parallel to the x-axis. The characteristic polynomial is

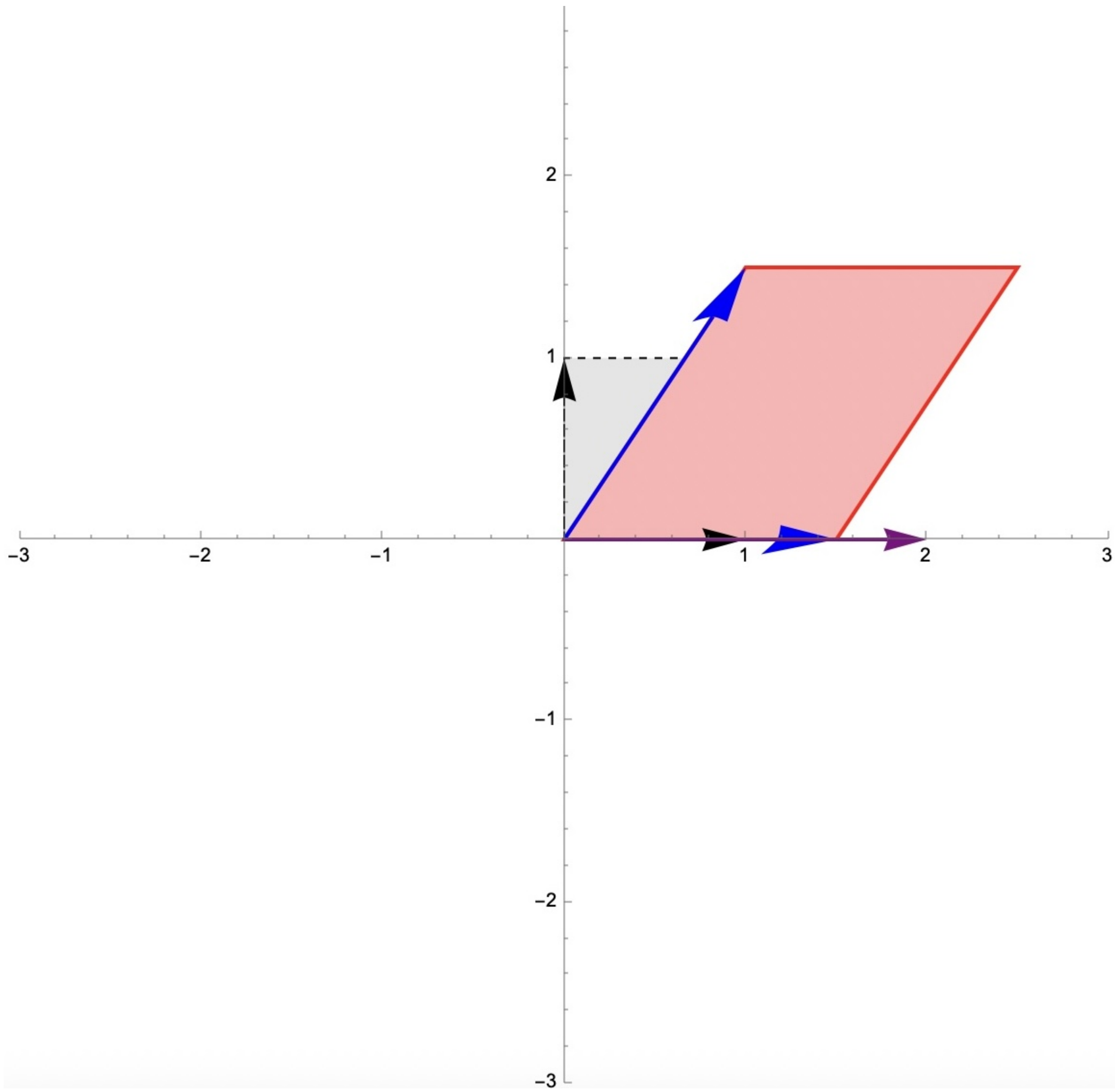
$$\det(J - tI) = (\lambda - t)^2 = 0,$$

so J has a single eigenvalue λ of algebraic multiplicity 2. However, there is only one linearly independent eigenvector:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad J\mathbf{v}_1 = \lambda \mathbf{v}_1.$$

Since there is only one eigenvector, the matrix J is not diagonalizable.

The gray square represents the original unit square in the plane. The red square shows the image of the unit square after transformation by J. The black dashed black arrows indicate the original basis vectors. The blue arrows show the transformed basis vectors. The purple arrow represents the eigenvector of J, which coincides with the coordinate axis.



-3
-2
-1
1
2
3
2
1
-1
-2
-3

Eigenvectors of linear transformations in $\mathbb{R}^3$

Now we consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

with respect to the standard basis of $\mathbb{R}^3$.

The unit cube spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is transformed into a parallelepiped spanned by $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$.

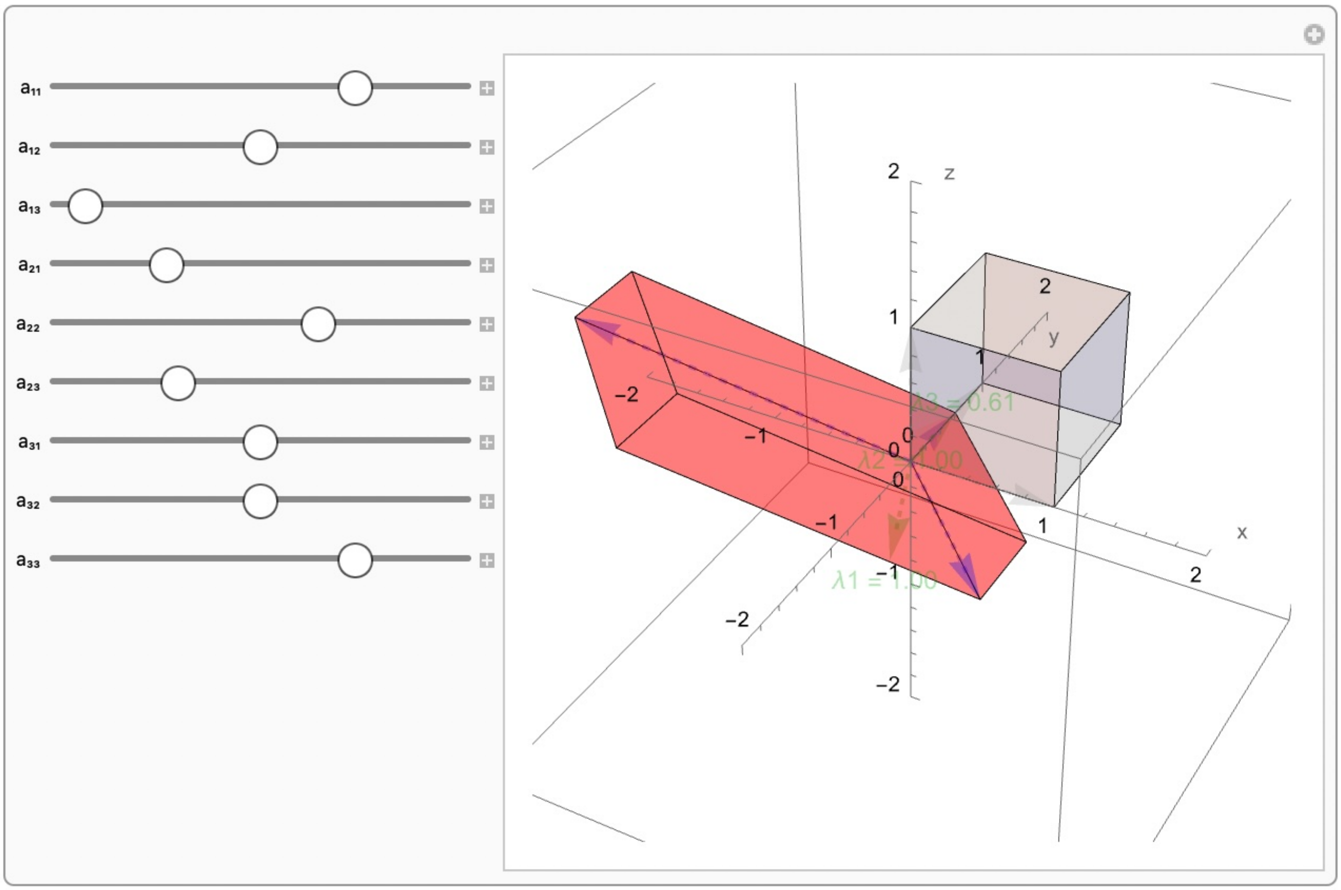
The gray cube corresponds to the unit cube before transformation.

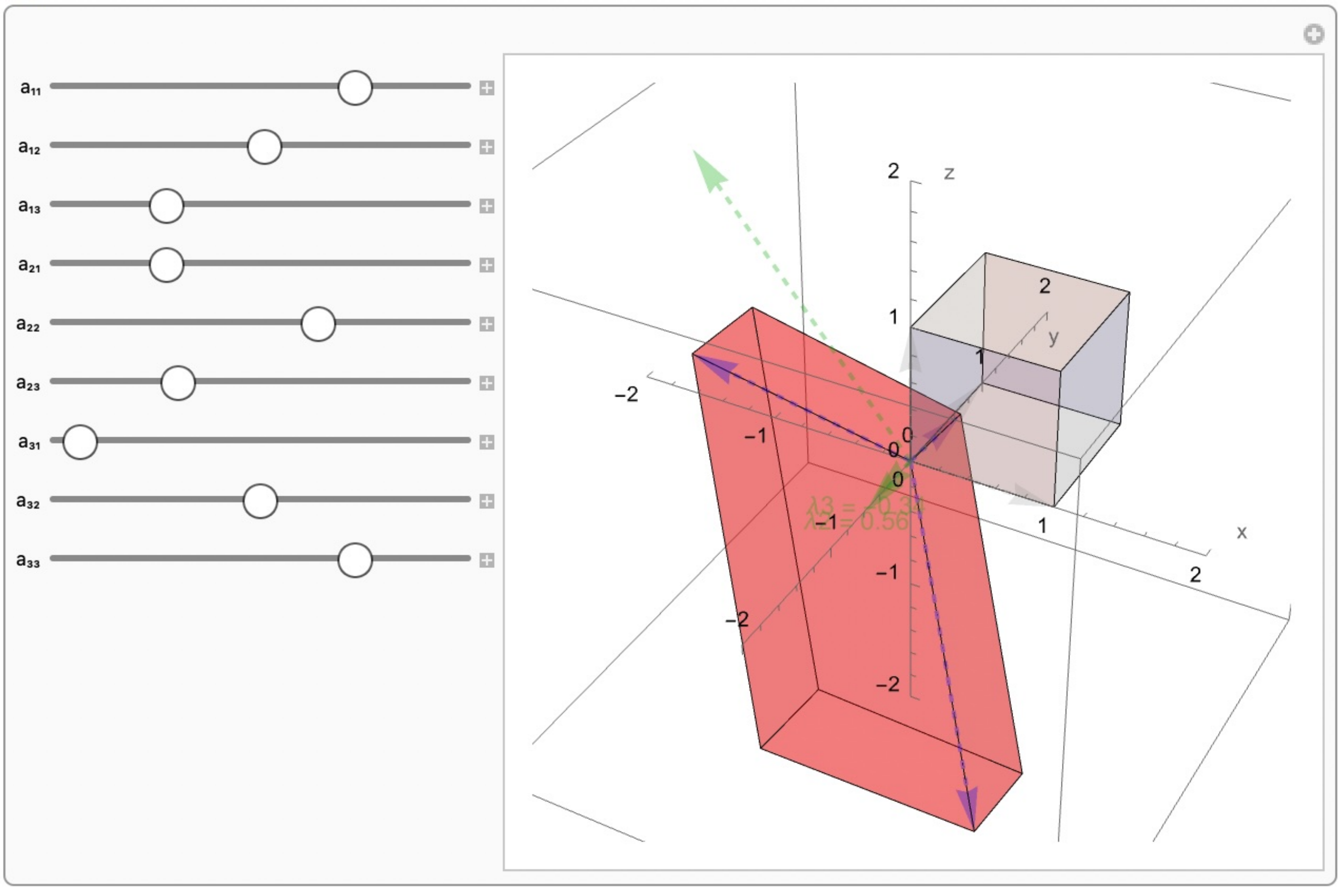
The dashed gray arrows indicate the original basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The red transparent cube shows the image of the unit cube under T, which is the parallelepiped spanned by the transformed basis vectors.

The blue arrows represent the transformed basis vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$.

The green arrows correspond to the eigenvectors, whose lengths reflect the magnitudes of their eigenvalues.


a_{11}
a_{12}
a_{13}
a_{21}
a_{22}
a_{23}
a_{31}
a_{32}
a_{33}
z
y
x
λ3 = 0.61


a11
a12
a13
a21
a22
a23
a31
a32
a33
x
y
z